

# Upper-bounding the $k$ -colorability threshold by counting covers

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## Abstract

Let  $G(n, m)$  be the random graph on  $n$  vertices with  $m$  edges. Let  $d = 2m/n$  be its average degree. We prove that  $G(n, m)$  fails to be  $k$ -colorable w.h.p. if  $d > 2k \ln k - \ln k - 1 + o_k(1)$ . This matches a conjecture put forward on the basis of sophisticated but non-rigorous statistical physics ideas (Krzakala, Pagnani, Weigt: Phys. Rev. E **70** (2004)). The proof is based on applying the first moment method to the number of “covers”, a physics-inspired concept. By comparison, a standard first moment over the number of  $k$ -colorings shows that  $G(n, m)$  is not  $k$ -colorable w.h.p. if  $d > 2k \ln k - \ln k$ .

*Key words:* random structures, phase transitions, graph coloring.

## 1 Introduction

Let  $G(n, m)$  be the random graph on  $V = \{1, \dots, n\}$  with  $m$  edges. Unless specified otherwise, we let  $m = \lceil dn/2 \rceil$  for a number  $d > 0$  that remains fixed as  $n \rightarrow \infty$ . Let  $k \geq 3$  be an  $n$ -independent integer. We say that  $G(n, m)$  has a property  $\mathcal{E}$  **with high probability** (‘w.h.p.’) if  $\lim_{n \rightarrow \infty} \mathbb{P}[G(n, m) \in \mathcal{E}] = 1$ .

One of the longest-standing open problems in the theory of random graphs is whether there is a phase transition for  $k$ -colorability in  $G(n, m)$  and, if so, at what average degree  $d$  it occurs [1, 10, 17]. Regarding existence, Achlioptas and Friedgut [1] proved that for any  $k \geq 3$  there is a *sharp threshold sequence*  $d_{k-\text{col}}(n)$  such that for any fixed  $\varepsilon > 0$  the random graph  $G(n, m)$  is  $k$ -colorable w.h.p. if  $m/n < (1 - \varepsilon)d_{k-\text{col}}(n)$  and non- $k$ -colorable w.h.p. if  $m/n > (1 + \varepsilon)d_{k-\text{col}}(n)$ . To establish the existence of an actual sharp threshold, one would have to show that the sequence  $d_{k-\text{col}}(n)$  converges. This is widely conjectured to be the case (explicitly so in [1]) but as yet unproven.

In any case, the techniques used to prove the existence of  $d_{k-\text{col}}(n)$  shed no light on its location. An upper bound is easily obtained via the *first moment method*. Indeed, a simple calculation shows that for  $k \geq 3$  and

$$d > d_{k,\text{first}} = 2k \ln k - \ln k, \quad (1)$$

the expected number of  $k$ -colorings tends to 0 as  $n \rightarrow \infty$  (e.g., [4]). Hence, Markov’s inequality implies that  $G(n, m)$  fails to be  $k$ -colorable for  $d > d_{k,\text{first}}$  w.h.p. Furthermore, Achlioptas and Naor [6] used the second moment method to prove that for any  $k \geq 3$ ,  $G(n, m)$  is  $k$ -colorable w.h.p. if

$$d_{k-\text{col}} \geq d_{k,\text{AN}} = 2(k-1) \ln(k-1) = 2k \ln k - 2 \ln k - 2 + o_k(1). \quad (2)$$

Here and throughout the paper, we use the symbol  $o_k(1)$  to hide terms that tend to zero for large  $k$ . The bound (2) was recently improved [12], also via a second moment argument, for sufficiently large  $k$  to

$$d_{k-\text{col}} \geq d_{k,\text{second}} = 2k \ln k - \ln k - 2 \ln 2 + o_k(1). \quad (3)$$

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This leaves an additive gap of  $2 \ln 2 + o_k(1)$  between the upper bound (1) and the lower bound (3).

The problem of  $k$ -coloring  $G(n, m)$  is closely related to the “diluted mean-field  $k$ -spin Potts antiferromagnet” model of statistical physics. Indeed, over the past decade physicists have developed sophisticated, albeit mathematically non-rigorous formalisms for identifying phase transitions in random discrete structures, the “replica method” and the “cavity method” (see [31] for details and references). Applied to the problem of  $k$ -coloring  $G(n, m)$  [27, 33, 34, 38], these techniques lead to the conjecture that

$$d_{k\text{-col}} = 2k \ln k - \ln k - 1 + o_k(1). \quad (4)$$

The main result of the present paper is an improved *upper* on  $d_{k\text{-col}}$  that matches the physics prediction (4) (at least up to the term hidden in the  $o_k(1)$ ).

**Theorem 1.1** *We have  $d_{k\text{-col}} \leq 2k \ln k - \ln k - 1 + o_k(1)$ .*

Theorem 1.1 improves the naive first moment bound (1) by about an additive 1. This proves, perhaps surprisingly, that the  $k$ -colorability threshold (if it exists) does *not* coincide with the first moment bound. Furthermore, Theorem 1.1 narrows the gap to the lower bound (3) to  $2 \ln 2 - 1 + o_k(1) \approx 0.39$ .

The proof of Theorem 1.1 is based on a concept borrowed from the “cavity method”, namely the notion of *covers*. This concept is closely related to hypotheses on the “geometry” of the set of  $k$ -colorings of the random graph, which are at the core of the cavity method [38, 31, 26, 27, 33, 37]. More precisely, let  $\mathcal{S}_k(G(n, m)) \subset \{1, \dots, k\}^n$  be the set of all  $k$ -colorings of  $G(n, m)$ . According to the cavity method, for average degrees  $(1 + o_k(1))k \ln k < d < d_{k\text{-col}}$  w.h.p. the set  $\mathcal{S}_k(G(n, m))$  has a decomposition

$$\mathcal{S}_k(G(n, m)) = \bigcup_{i=1}^N \mathcal{C}_i$$

into  $N = \exp(\Omega(n))$  non-empty “clusters”  $\mathcal{C}_i$  such that for any two colorings  $\sigma, \tau$  that belong to distinct clusters we have

$$\text{dist}(\sigma, \tau) = |\{v \in V : \sigma(v) \neq \tau(v)\}| \geq \delta n \quad \text{for some } \delta = \delta(k, d) > 0.$$

In other words, the clusters are well-separated. Furthermore, a “typical” cluster  $\mathcal{C}_i$  is characterized by a set of  $\Omega(n)$  “frozen” vertices, which have the same color in all colorings  $\sigma \in \mathcal{C}_i$ . Roughly speaking, a cover is a representation of a cluster  $\mathcal{C}_i$ : the cover details the colors of all the frozen vertices, while the non-frozen ones are represented by the “joker color” 0. We will define covers precisely in Section 3.

The key idea behind the proof of Theorem 1.1 is to apply the first moment method to the number of covers. Since, according to the cavity method, covers are in one-to-one correspondence with clusters, we carry effectively out a first moment argument for the number of *clusters*. The improvement over the “classical” first moment bound for the number of  $k$ -colorings results because this approach allows us to completely ignore the cluster sizes  $|\mathcal{C}_i|$ . Indeed, close to the  $k$ -colorability threshold the cluster sizes are conjectured to vary wildly, as has in part been established rigorously in [12]. By contrast, the “classical” first moment argument amounts to putting a rather generous uniform bound on all the cluster sizes.

The “freezing” of vertices in  $k$ -colorings of  $G(n, m)$  has been studied previously [2]. Formally, let us call a set  $F$  of vertices  $\delta$ -**frozen** in a  $k$ -coloring  $\sigma$  of  $G(n, m)$  if any other  $k$ -coloring  $\tau$  such that  $\tau(v) \neq \sigma(v)$  for some vertex  $v \in F$  indeed satisfies  $|\{v \in F : \sigma(v) \neq \tau(v)\}| \geq \delta n$ . There is an explicitly known sharp threshold  $d_{k, \text{freeze}} = (1 + o_k(1))k \ln k$ , about half of  $d_{k\text{-col}}$ , such that for  $d > d_{k, \text{freeze}}$  w.h.p. a random  $k$ -coloring of  $G(n, m)$  has  $\Omega(n)$  frozen vertices [32]. The threshold  $d_{k, \text{freeze}}$  coincides asymptotically with the largest average degree for which efficient algorithms are known to find a  $k$ -coloring of  $G(n, m)$  w.h.p. [3, 21]. In fact, it has been hypothesized that the emergence of frozen vertices causes the failure of a wide class of “local search” algorithms [2, 32].

Yet the known results [2, 32] on the freezing phenomenon only show that a *random*  $k$ -coloring of  $G(n, m)$  “freezes”. It is not apparent that this poses an obstacle if we merely aim to find *some*  $k$ -coloring. As an important part of the proof of Theorem 1.1, we show that for  $d$  close to  $d_{k\text{-col}}$  (but strictly below the lower bound (3)), in fact *all*  $k$ -colorings of  $G(n, m)$  belong to a cluster with many frozen vertices w.h.p.

**Corollary 1.2** Assume that  $d \geq 2k \ln k - \ln k - 4 + o_k(1)$ . There is a number  $\delta_k > 0$  such that w.h.p. every  $k$ -coloring  $\sigma$  of the random graph  $G(n, m)$  has a set  $F(\sigma)$  of  $\delta_k$ -frozen vertices of size  $|F(\sigma)| \geq (1 - o_k(1))n$ .

Due to the (conjectured) relationship between freezing and the demise of local-search algorithms, it would be interesting to identify the precise threshold where *all* the  $k$ -colorings of  $G(n, m)$  are frozen.

**Further related work.** The problem of coloring  $G(n, m)$  has been studied intensively over the past few decades. Improving a prior result by Matula [30], Bollobás [9] determined the asymptotic value of the chromatic number of dense random graphs. Łuczak extended this result to sparse random graphs [28]. In the case that  $d$  remains fixed as  $n \rightarrow \infty$ , his result yields  $d_{k\text{-col}} = (2 + o_k(1))k \ln k$ . As mentioned above, Achlioptas and Naor [6] improved this result by obtaining the lower bound (2). In addition, Łuczak’s result was sharpened in [11] for  $m \ll n^{5/4}$ .

The problem of locating the threshold for 3-colorability has received considerable attention as well. The best current lower bound is 4.03 [5]. Moreover, Dubois and Mandler [14] proved that  $d_{3\text{-col}} \leq 4.9364$ . This improved over a stream of prior results [4, 15, 18, 20, 24].

The key idea in this line of work is to estimate the first moment of the number of “rigid” colorings: for any two colors  $1 \leq i < j \leq k$ , every vertex of color  $i$  must have neighbors of color  $j$  [4]. Clearly, any  $k$ -colorable graph must have a rigid  $k$ -coloring. At the same time, the number of rigid  $k$ -colorings can be expected to be significantly smaller than the total number of  $k$ -colorings, and thus one might expect an improved first-moment upper bound. However, in terms of the clustering scenario put forward by physicists, it is conceivable that many clusters contain a large (in fact, exponentially large) number of rigid  $k$ -colorings. Therefore, the idea of counting rigid  $k$ -colorings seems conceptually weaker than the approach of counting clusters pursued in the present work. In fact, the improvement obtained by counting rigid colorings appears to diminish for larger  $k$  [4].

A fairly new approach to obtaining upper bounds on thresholds in random constraint satisfaction problems is the use of the *interpolation method* [8, 19, 22, 35]. This technique gives an upper bound on, e.g., the  $k$ -colorability threshold in terms of a variational problem that is related to the statistical mechanics techniques. However, this variational problem appears to be difficult to solve. Thus, it is not clear (to me) how an *explicit* upper bound as stated in Theorem 1.1 can be obtained from the interpolation method.

Dani, Moore and Olson [13] studied a variant of the graph coloring problem in which each pair of  $(u, v)$  of vertices comes with a random permutation  $\pi_{u,v}$  of the  $k$  possible colors; this gives rise to a concept of “permuted”  $k$ -colorings. They obtained an upper bound of  $2k \ln k - \ln k - 1 + o_k(1)$  on the threshold for the existence of permuted  $k$ -colorings. The proof is based on counting the total weight of  $k$ -colorings and using an isoperimetric inequality. Moreover, as pointed out in [13], physics intuition suggests that the threshold in the permuted  $k$ -coloring problem matches the “unpermuted”  $k$ -colorability threshold.

In the context of satisfiability, Maneva and Sinclair [29] used the concept of covers to obtain a conditional upper bound on the random 3-SAT threshold. Roughly speaking, the condition that they need is that w.h.p. all satisfying assignments have frozen variables. However, verifying this condition in random 3-SAT is an open problem. (That said, it is conceivable that the approach used in [29] might yield a better upper bound on the  $k$ -SAT threshold for large  $k$ .)

## 2 Preliminaries

Let  $[k] = \{1, 2, \dots, k\}$ . Because Theorem 1.1 and Corollary 1.2 are asymptotic statements in both  $k$  and  $n$ , we may generally assume that  $k \geq k_0$  and  $n \geq n_0$ , where  $k_0, n_0$  are constants that are chosen sufficiently large for the various estimates to hold.

We perform asymptotic considerations with respect to both  $k$  and  $n$ . When referring to asymptotics in  $k$ , we use the notation  $O_k(\cdot)$ ,  $o_k(\cdot)$ , etc. Asymptotics with respect to  $n$  are just denoted by  $O(\cdot)$ ,  $o(\cdot)$ , etc.

If  $G$  is a (multi-)graph and  $A, B$  are sets of vertices, then we let  $e_G(A, B)$  denote the number of  $A$ - $B$ -edges in  $G$ . Moreover,  $e_G(A)$  denotes the number of edges inside of  $A$ . If  $A = \{v\}$  is a singleton, we just write  $e_G(v, B)$ . The reference to  $G$  is omitted where it is clear from the context.

**Working with independent edges.** The random graph  $G(n, m)$  consists of  $m$  edges that are chosen *almost* independently. To simplify some of the arguments below, we are going to work with a random multi-graph model  $G'(n, m)$  in which edges are perfectly independent. More precisely,  $G'(n, m)$  is obtained as follows: let  $e = (e_1, \dots, e_m) \in (V \times V)^m$  be a uniformly random  $m$ -tuple of ordered pairs of vertices. In other words, each  $e_i$  is chosen uniformly out of all  $n^2$  possible vertex pairs, independently of all the others. Now, let  $G'(n, m)$  be the random multi-graph comprising of  $e_1, \dots, e_m$  viewed as undirected edges. Thus,  $G'(n, m)$  may have self-loops (if  $e_i = (v, v)$  for some index  $i$ ) as well as multiple edges (if, for example,  $e_i = (u, v)$  and  $e_j = (v, u)$  with  $1 \leq i < j \leq m$  and  $u \neq v$ ). The two random graph models are related as follows.

**Lemma 2.1** *For any event  $\mathcal{A}$  we have  $P[G(n, m) \in \mathcal{A}] \leq O(1) \cdot P[G'(n, m) \in \mathcal{A}]$ .*

*Proof.* The random graph  $G'(n, m)$  has *at most*  $m$  distinct edges, and no self-loops. Let  $\mathcal{E}$  be the event that it has *exactly*  $m$  edges. This is the case iff  $e_1, \dots, e_m$  induce pairwise distinct undirected edges. Given the event  $\mathcal{E}$ ,  $G'(n, m)$  is identical to  $G(n, m)$ . Hence,

$$P[G(n, m) \in \mathcal{A}] = P[G'(n, m) \in \mathcal{A} | \mathcal{E}] \leq P[G'(n, m) \in \mathcal{A}] / P[\mathcal{E}] \quad (5)$$

Now,

$$P[\mathcal{E}] \geq \prod_{i=0}^m \left(1 - \frac{2i + n}{n^2}\right) = \exp \left[ \sum_{i=0}^{m-1} \ln(1 - (2i + n)/n^2) \right] \geq \exp(-d - 2d^2) = \Omega(1).$$

Thus, the assertion follows from (5).  $\square$

**The Chernoff bound.** We need the following Chernoff bound on the tails of a binomially distributed random variable (e.g., [23, p. 21]).

**Lemma 2.2** *Let  $\varphi(x) = (1 + x) \ln(1 + x) - x$ . Let  $X$  be a binomial random variable with mean  $\mu > 0$ . Then for any  $t > 0$  we have*

$$\begin{aligned} P[X > E[X] + t] &\leq \exp(-\mu \cdot \varphi(t/\mu)), \\ P[X < E[X] - t] &\leq \exp(-\mu \cdot \varphi(-t/\mu)). \end{aligned}$$

*In particular, for any  $t > 1$  we have  $P[X > t\mu] \leq \exp[-t\mu \ln(t/e)]$ .*

**Balls and bins.** Consider a balls and bins experiment where  $\mu$  balls are thrown independently and uniformly at random into  $\nu$  bins. Thus, the probability of each distribution of balls into bins equals  $\nu^{-\mu}$ . We will need the following well-known ‘‘Poissonization lemma’’ (e.g., [16, Section 2.6]).

**Lemma 2.3** *In the above experiment let  $e_i$  be the number of balls in bin  $i \in [\nu]$ . Moreover, let  $\lambda > 0$  and let  $(b_i)_{i \in [\nu]}$  be a family of independent Poisson variables, each with mean  $\lambda$ . Then for any sequence  $(t_i)_{i \in [\nu]}$  of non-negative integers such that  $\sum_{i=1}^{\nu} t_i = \mu$  we have*

$$P[\forall i \in [\nu] : e_i = t_i] = P\left[\forall i \in [\nu] : b_i = t_i \mid \sum_{i=1}^{\nu} b_i = \mu\right].$$

*Hence, the joint distribution of  $(e_i)_{i \in [\nu]}$  coincides with the joint distribution of  $(b_i)_{i \in [\nu]}$  given  $\sum_{i=1}^{\nu} b_i = \mu$ .*

We are typically going to use Lemma 2.3 to obtain an *upper* bound on the probability on the left hand side. Therefore, the following simple corollary will come in handy.

**Corollary 2.4** *With the notation of Lemma 2.3, assume that  $\lambda = \mu/\nu > 0$ . Then for any sequence  $(t_i)_{i \in [\nu]}$  of non-negative integers such that  $\sum_{i=1}^{\nu} t_i = \mu$  we have*

$$P[\forall i \in [\nu] : e_i = t_i] \leq O(\sqrt{\mu}) \cdot P[\forall i \in [\nu] : b_i = t_i].$$

*Proof.* Let  $b = \sum_{i=1}^{\nu} b_i = \mu$ . Since the  $b_i$  are independent Poisson variables with means  $\lambda = \mu/\nu$ ,  $b$  is Poisson with mean  $\mu$ . By Stirling's formula,  $P[b = \mu] = \mu^\mu \exp(-\mu)/\mu! = \Omega(\mu^{-1/2})$ . Hence, Lemma 2.3 yields

$$P[\forall i \in [\nu] : e_i = t_i] = \frac{P[\forall i \in [\nu] : b_i = t_i]}{P[b = \mu]} = O(\sqrt{\mu}) \cdot P[\forall i \in [\nu] : b_i = t_i],$$

as claimed.  $\square$

### 3 Covers

Let  $G = (V, E)$  be a graph, let  $k \geq k_0$  be an integer, and let  $\sigma : V \rightarrow [k]$  be a  $k$ -coloring of  $G$ . We would like to identify a set  $F \subset V$  of vertices whose colors cannot be changed easily by a “local” recoloring of a few vertices. For instance, if  $v$  is a vertex that does not have a neighbor of color  $j$  for some  $j \in [k] \setminus \{\sigma(v)\}$ , then  $v$  can be recolored easily. More generally, we would like to say that, recursively, a vertex can be recolored easily if there is a color  $j$  such that all its neighbors of color  $j$  can be easily recolored. To formalize this, we need the following concept.

**Definition 3.1** Let  $\zeta : V \rightarrow \{0, 1, \dots, k\}$ . We call  $v \in V$  **stable** under  $\zeta$  if  $\zeta(v) \neq 0$  and if for any color  $j \in [k] \setminus \{\zeta(v)\}$  there are at least two neighbors  $u_1, u_2$  of  $v$  such that  $\zeta(u_1) = \zeta(u_2) = j$ .

Now, consider the following **whitening process** that, given a  $k$ -coloring  $\sigma$  of  $G$ , returns a map  $\hat{\sigma} : V \rightarrow \{0, 1, \dots, k\}$ ; the idea is that  $\hat{\sigma}(v) = 0$  for all  $v$  that are easy to recolor.

**WH1.** Initially, let  $\hat{\sigma}(v) = \sigma(v)$  for all  $v \in V$ .

**WH2.** While there exist a vertex  $v \in V$  with  $\hat{\sigma}(v) \neq 0$  that is not stable under  $\hat{\sigma}$ , set  $\hat{\sigma}(v) = 0$ .

The process **WH1–WH2** is similar to processes studied in [36, 37] in the context of random graph coloring, and in [7] in the context of random  $k$ -SAT. (The term “whitening process” stems from [36].) Clearly, the final outcome  $\hat{\sigma}$  of the whitening process is independent of the order in which **WH2** proceeds.

The intuition behind the whitening process is that if we attempt to recolor some stable vertex  $v$  with another color  $j \in [k] \setminus \{\hat{\sigma}(v)\}$ , then we will have to recolor *two* additional stable vertices  $u_1, u_2$  as well. Hence, any attempt to recolor a stable vertex is liable to trigger an avalanche of further recolorings (unless the graph  $G$  has an abundance of short cycles, which is well-known not to be the case in the random graph  $G(n, m)$  w.h.p.).

The following definition is going to lead to a neat description of the outcome of the whitening process.

**Definition 3.2** A  $k$ -**cover** in  $G$  is a map  $\zeta : V \rightarrow \{0, 1, \dots, k\}$  with the following properties.

**CV1.** There is no edge  $e = \{u, v\}$  such that  $\zeta(u) = \zeta(v) \neq 0$ .

**CV2.** If  $\zeta(v) \neq 0$ , then  $v$  is stable under  $\zeta$ .

**CV3.** If  $\zeta(v) = 0$ , then there are  $i, j \in [k]$ ,  $i \neq j$ , such that  $v$  does not have a neighbor  $u$  with  $\zeta(u) = i$  and  $v$  has at most one neighbor  $w$  with  $\zeta(w) = j$ .

The concept of covers is very closely related and, in fact, inspired by the properties of certain fixed points of the Survey Propagation message passing procedure [31]. (To my knowledge, the term “cover” has not been used previously in the context of  $k$ -colorability, although it appears to be in common use in the context of satisfiability [29].)

Now, the outcome of  $\hat{\sigma}$  is the cover characterized by the following two properties.

- i. For all vertices  $v$  such that  $\hat{\sigma}(v) \neq 0$  we have  $\hat{\sigma}(v) = \sigma(v)$ .
- ii. Subject to i.,  $|\hat{\sigma}^{-1}(0)|$  is minimum.

Of course, in general the graph  $G$  may have many  $k$ -covers that cannot be obtained from a  $k$ -coloring via the whitening process. This motivates

**Definition 3.3** A  $k$ -cover  $\zeta$  of  $G$  is *valid* if  $G$  has a  $k$ -coloring  $\sigma$  such that  $\zeta = \hat{\sigma}$ .

To prove Theorem 1.1, we perform a first moment argument for the number of valid  $k$ -covers. The main task is to show that the all-0 cover (i.e.,  $\zeta(v) = 0$  for all vertices  $v$ ) is not a valid  $k$ -cover in  $G(n, m)$  w.h.p. To this end, we need to establish a few basic properties that all  $k$ -colorings of  $G(n, m)$  have w.h.p. More precisely, in Section 4 we are going to prove the following via a “standard” first moment argument over  $k$ -colorings.

**Proposition 3.4** Assume that  $k \geq k_0$  for a sufficiently large constant  $k_0$ . Moreover, assume that  $d = 2k \ln k - \ln k - c$ , with  $0 \leq c \leq 4$ .

1. Let  $Z$  be the number of  $k$ -colorings of  $G(n, m)$ . Then  $\frac{1}{n} \ln \mathbb{E}[Z] = \frac{c + o_k(1)}{2k}$ .
2. W.h.p. all  $k$ -colorings of  $G(n, m)$  satisfy  $|\sigma^{-1}(i)| = (1 + o_k(1))\frac{n}{k}$  for all  $i \in [k]$ .
3. In fact, w.h.p.  $G(n, m)$  does not have a  $k$ -coloring  $\sigma$  such that  $|\sigma^{-1}(i) - n/k| > n/(k \ln^4 k)$  for more than  $\ln^8 k$  colors  $i \in [k]$ .

Building upon Proposition 3.4, we will establish the following properties of valid  $k$ -covers in Section 5.

**Proposition 3.5** There is a number  $k_0$  such that for  $k \geq k_0$  and  $2k \ln k - \ln k - 4 \leq d \leq 2k \ln k$  any valid  $k$ -cover  $\zeta$  of  $G(n, m)$  has the following properties w.h.p.

1. We have  $|\zeta^{-1}(0)| \leq nk^{-2/3}$ .
2. For all  $i \in [k]$  we have  $|\zeta^{-1}(i)| = (1 + o_k(1))n/k$ .
3. In fact, there are no more than  $\ln^9 k$  indices  $i \in [k]$  such that  $|\zeta^{-1}(i) - n/k| > n/(k \ln^3 k)$ .

Finally, in Section 6 we perform the first moment argument over  $k$ -covers.

**Proposition 3.6** There is  $\varepsilon_k = o_k(1)$  such that for  $d \geq 2k \ln k - \ln k - 1 + \varepsilon_k$  w.h.p. the random graph  $G(n, m)$  does not have a  $k$ -cover with properties 1.–3. from Proposition 3.5.

Theorem 1.1 is immediate from Propositions 3.5 and 3.6. Furthermore, we will prove Corollary 1.2 in Section 5.

## 4 Proof of Proposition 3.4

The proof of Proposition 3.4 is very much based on standard arguments, reminiscent but unfortunately not (quite) identical to estimates from, e.g., [4]. Suppose  $d = 2k \ln k - \ln k - c$  with  $0 \leq c \leq 4$ . Throughout this section we work with the random graph  $G'(n, m)$  with  $m$  independent edges.

**Lemma 4.1** Let  $\nu = (\nu_1, \dots, \nu_k)$  be a  $k$ -tuple of non-negative integers such that  $\sum_{i=1}^k \nu_i = n$ . Let  $Z_\nu$  be the number of  $k$ -colorings  $\sigma$  of  $G'(n, m)$  such that  $|\sigma^{-1}(i)| = \nu_i$  for all  $i \in [k]$ . Then

$$\ln \mathbb{E}[Z_\nu] = o(n) + \sum_{i=1}^k \nu_i \ln(n/\nu_i) + \frac{d}{2} \ln \left[ 1 - \sum_{i=1}^k \left( \frac{\nu_i}{n} \right)^2 \right]. \quad (6)$$



*Proof.* Let  $\Sigma_\nu$  be the set of all  $\sigma : V \rightarrow [k]$  such that  $|\sigma^{-1}(i)| = \nu_i$  for all  $i \in [k]$ . By Stirling's formula,

$$\ln |\Sigma_\nu| = o(n) + \sum_{i=1}^k \nu_i \ln(n/\nu_i). \quad (7)$$

Furthermore, the probability of being a  $k$ -coloring in  $G'(n, m)$  is the same for all  $\sigma \in \Sigma_\nu$ . In fact, due to the independence of the edges in  $G'(n, m)$ , this probability is  $q = (1 - \sum_{i=1}^k (\nu_i/n)^2)^m$ , because  $\sigma$  is a  $k$ -coloring iff each of the color classes  $\sigma^{-1}(i)$  is an independent set. As  $\mathbb{E}[Z_\nu] = |\Sigma_\nu| \cdot q$ , the assertion follows from (7).  $\square$

**Corollary 4.2** *Let  $Z$  be the total number of  $k$ -colorings of  $G'(n, m)$ . We have*

$$\frac{1}{n} \ln \mathbb{E}[Z] = \ln k + \frac{d}{2} \ln(1 - 1/k) + o(1) = \frac{c}{2k} + O_k(\ln k/k^2).$$

*Proof.* Let  $\mathcal{N}$  be the set of all  $k$ -tuples  $\nu = (\nu_1, \dots, \nu_k)$  of non-negative integers such that  $\sum_{i=1}^k \nu_i = n$ . Then  $\mathbb{E}[Z] = \sum_{\nu \in \mathcal{N}} \mathbb{E}[Z_\nu] \leq n^k \max_{\nu \in \mathcal{N}} \mathbb{E}[Z_\nu]$ . Hence, Lemma 4.1 yields

$$\frac{1}{n} \ln \mathbb{E}[Z] = o(1) + \max \left\{ \sum_{i=1}^k \nu_i \ln(n/\nu_i) + \frac{d}{2} \ln \left[ 1 - \sum_{i=1}^k \left( \frac{\nu_i}{n} \right)^2 \right] : \nu \in \mathcal{N} \right\}. \quad (8)$$

Letting  $\mathcal{A}$  be the set of all  $k$ -tuples  $\alpha = (\alpha_1, \dots, \alpha_k) \in [0, 1]^k$  such that  $\sum_{i=1}^k \alpha_i = 1$ , we obtain from (8)

$$\frac{1}{n} \ln \mathbb{E}[Z] = o(1) + \max \left\{ -\sum_{i=1}^k \alpha_i \ln(\alpha_i) + \frac{d}{2} \ln \left[ 1 - \sum_{i=1}^k \alpha_i^2 \right] : \alpha \in \mathcal{A} \right\}. \quad (9)$$

The entropy function  $-\sum_{i=1}^k \alpha_i \ln(\alpha_i)$  is well-known to attain its maximum at the point  $\alpha = \frac{1}{k} \mathbf{1}$  with all  $k$  entries equal to  $1/k$ . Furthermore, the sum of squares  $\sum_{i=1}^k \alpha_i^2$  attains its minimum at  $\alpha = \frac{1}{k} \mathbf{1}$  as well. Hence, the term  $\frac{d}{2} \ln[1 - \sum_{i=1}^k \alpha_i^2]$ , and thus (9), is maximized at  $\frac{1}{k} \mathbf{1}$ . Consequently,

$$\begin{aligned} \frac{1}{n} \ln \mathbb{E}[Z] &= \ln k + \frac{d}{2} \ln(1 - 1/k) + o(1) = \ln k - \frac{d}{2} \left[ \frac{1}{k} + \frac{1}{2k^2} + O_k(k^{-3}) \right] \\ &= \ln k - \left[ k \ln k - \frac{\ln k}{2} - \frac{c}{2} \right] \cdot \left[ \frac{1}{k} + \frac{1}{2k^2} + O_k(k^{-3}) \right] = \frac{c}{2k} + O(\ln k/k^2), \end{aligned}$$

as claimed.  $\square$

**Corollary 4.3** *W.h.p. all  $k$ -colorings  $\sigma$  of  $G'(n, m)$  satisfy  $|\sigma^{-1}(i)| = (1 + o_k(1)) \frac{n}{k}$  for all  $i \in [k]$ , and there is no  $k$ -coloring  $\sigma$  such that  $|\sigma^{-1}(i) - 1/k| > 1/(k \ln^4 k)$  for more than  $\ln^8 k$  colors  $i \in [k]$ .*

*Proof.* Let  $\nu = (\nu_1, \dots, \nu_k)$  be a  $k$ -tuple of non-negative integers such that  $\sum_{i=1}^k \nu_i = n$ . We are going to estimate  $\mathbb{E}[Z_\nu]$  in terms of how much  $\nu$  deviates from the “flat” vector  $(n/k, \dots, n/k)$ . To this end, we compute the first two differentials of (6). Set  $\alpha = (\alpha_1, \dots, \alpha_k) = \nu/n$  and let  $f(\alpha) = -\sum_{i=1}^k \alpha_i \ln \alpha_i + \frac{d}{2} \ln(1 - \sum_{i=1}^k \alpha_i^2)$ . Since  $\sum_{i=1}^k \alpha_i = 1$ , we can eliminate the variable  $\alpha_k = 1 - \sum_{i=1}^{k-1} \alpha_i$ . Hence, we obtain for  $i, j \in [k-1]$ ,  $i \neq j$

$$\begin{aligned} \frac{\partial f}{\partial \alpha_i} &= \ln(\alpha_k/\alpha_i) + \frac{d(\alpha_k - \alpha_i)}{1 - \|\alpha\|_2^2}, \\ \frac{\partial^2 f}{\partial \alpha_i^2} &= -\frac{1}{\alpha_k} - \frac{1}{\alpha_i} - \frac{2d}{1 - \|\alpha\|_2^2} - \frac{2d(\alpha_k - \alpha_i)^2}{(1 - \|\alpha\|_2^2)^2}, \end{aligned} \quad (10)$$

$$\frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j} = -\frac{1}{\alpha_k} - \frac{d}{1 - \|\alpha\|_2^2} - \frac{2d(\alpha_k - \alpha_i)(\alpha_k - \alpha_j)}{(1 - \|\alpha\|_2^2)^2}. \quad (11)$$

In particular, the first differential  $Df$  vanishes at  $\alpha = \frac{1}{k}\mathbf{1}$ . At this point, the Hessian  $D^2f = (\frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j})_{i,j \in [k-1]}$  is negative-definite, whence  $\alpha = \frac{1}{k}\mathbf{1}$  is a local maximum. Because the rank-one matrix  $((\alpha_k - \alpha_i) \cdot (\alpha_k - \alpha_j))_{i,j \in [k-1]}$  is positive semidefinite for all  $\alpha$ , (10) and (11) show that  $D^2f$  is negative-definite for all  $\alpha$ . In fact, due to the  $-\frac{2d}{1-\|\alpha\|_2^2}$  term in (10), all its eigenvalues are smaller than  $-\frac{d}{1-\|\alpha\|_2^2} \leq -d$ . Therefore, Taylor's theorem yields that

$$f(\alpha) \leq f(k^{-1}\mathbf{1}) - \frac{d}{2} \|\alpha - k^{-1}\mathbf{1}\|_2^2$$

for all  $\alpha$ . Hence, Corollary 4.2 implies that

$$\frac{1}{n} \ln \mathbb{E}[Z_\nu] \leq \frac{c}{2k} + O_k(\ln k/k^2) - \frac{d}{2} \|\alpha - k^{-1}\mathbf{1}\|_2^2. \quad (12)$$

Since  $d = (2 - o_k(1))k \ln k$ , the right hand side of (12) is negative if either

- $\max_{i \in [k]} |\alpha_i - k^{-1}| > (k \ln^{1/3} k)^{-1}$ , or
- there are more than  $\ln^8 k$  indices  $i \in [k]$  such that  $|\alpha_i - 1/k| > (k \ln^4 k)^{-1}$ .

Thus, Markov's inequality shows that w.h.p. there is no  $k$ -coloring with either of these properties.  $\square$

Finally, Proposition 3.4 is immediate from Lemma 2.1 and Corollaries 4.2 and 4.3.

## 5 Proof of Proposition 3.5

Suppose  $d = 2k \ln k - \ln k - c$  with  $0 \leq c \leq 4$ . Throughout this section we work with the random graph  $G'(n, m)$  with  $m$  independent edges.

### 5.1 The core

Let  $\sigma : V \rightarrow [k]$  be a map such that  $|\sigma^{-1}(i)| = (1 + o_k(1))\frac{n}{k}$  for all  $i \in [k]$ . Moreover, let  $G'(\sigma)$  be the random multi-graph  $G'(n, m)$  conditional on  $\sigma$  being a valid  $k$ -coloring. Thus,  $G'(\sigma)$  consists of  $m$  independent random edges  $e_1 = (u_1, v_1), \dots, e_m = (u_m, v_m)$  such that  $\sigma(u_i) \neq \sigma(v_i)$  for all  $i \in [m]$ . To prove Proposition 3.5 we need to show that with a very high probability, a large number of vertices of  $G'(\sigma)$  will remain “unscathed” by the whitening process **WH1–WH2**. To exhibit such vertices, we consider the following construction. Let  $\ell = \exp(-7) \ln k$  and assume that  $k \geq k_0$  is large enough so that  $\ell > 3$ . Let  $V_i = \sigma^{-1}(i)$  for  $i \in [k]$ .

**CR1** For  $i \in [k]$  let  $W_i = \{v \in V_i : \exists j \neq i : e(v, V_j) < 3\ell\}$  and  $W = \bigcup_{i=1}^k W_i$ .

**CR2** Let  $U = \{v \in V : \exists j : e(v, W_j) > \ell\}$ .

**CR3** Set  $Y = U$ . While there is a vertex  $v \in V \setminus Y$  that has  $\ell$  or more neighbors in  $Y$ , add  $v$  to  $Y$ .

We call the graph  $G'(\sigma) - W - Y$  obtained by removing the vertices in  $W \cup Y$  the **core** of  $G'(\sigma)$ . By construction, every vertex  $v$  in the core has at least  $\ell$  neighbors of each color  $j \neq \sigma(v)$  that also belong to the core. In effect, if  $\hat{\sigma}$  is the outcome of the whitening process applied to  $G'(\sigma)$ , then  $\hat{\sigma}(v) = \sigma(v)$  for all vertices  $v$  in the core.

The construction **CR1–CR3** has been considered previously to show that a *random*  $k$ -coloring of the random graph  $G(n, m)$  has many frozen vertices w.h.p. [2, 12]. In the present context we need to perform a rather more thorough analysis of the process **CR1–CR3** to show that w.h.p. *all*  $k$ -colorings  $\sigma$  of  $G(n, m)$  induce a non-zero cover  $\hat{\sigma}$ . To obtain such a strong result, we need to control the large deviations of various quantities, particularly the sizes of the sets  $W$ ,  $W_i$  and  $U$ . More precisely, in Section 5.2 we prove

**Lemma 5.1** *With probability at least  $1 - \exp(-16n/k)$  the random graph  $G'(\sigma)$  has the following properties.*



1. We have  $|W| \leq nk^{-0.7}$ .
2. For all  $i \in [k]$  we have  $|W_i| \leq \frac{n \ln \ln k}{k \ln k}$ .
3. There are no more than  $\ln^4 k$  indices  $i \in [k]$  such that  $|W_i| \geq \frac{n}{k \ln^4 k}$ .

Moreover, in Section 5.3 we are going to establish

**Lemma 5.2** In  $G'(\sigma)$  we have  $P[|U| > \frac{n \ln \ln k}{k \ln k}] \leq \exp(-10n/k)$ .

To estimate the size of  $Y$  we use the following observation.

**Lemma 5.3** W.h.p. the random graph  $G'(n, m)$  has the following property.

$$\text{For any set } \mathcal{Y} \subset V \text{ of size } |\mathcal{Y}| \leq \lceil \frac{2n \ln \ln k}{k \ln k} \rceil \text{ we have } e(\mathcal{Y}) < \frac{\ell}{2} |\mathcal{Y}| \quad (13)$$

*Proof.* For any fixed set  $\mathcal{Y}$  of size  $0 < yn \leq \lceil \frac{2n \ln \ln k}{k \ln k} \rceil$  the number  $e(\mathcal{Y})$  of edges spanned by  $\mathcal{Y}$  in  $G'(n, m)$  is binomially distributed with mean

$$E[e(\mathcal{Y})] = \frac{m(yn)^2}{n^2} = (1 + o_k(1))y^2 dn/2 \leq 2y^2 nk \ln k$$

Hence, by the Chernoff bound

$$P[e(\mathcal{Y}) \geq y\ell n/2] \leq \exp\left[\frac{y\ell n}{2} \ln\left(\frac{y\ell n/2}{e \cdot E[e(\mathcal{Y})]}\right)\right] \leq \exp\left[\frac{y\ell n}{3} \ln(ky)\right]. \quad (14)$$

Since  $ky \leq 3 \ln \ln k / \ln k$  and  $\ell = \Omega_k(\ln k)$ , (14) yields

$$P[e(\mathcal{Y}) \geq y\ell n/2] \leq \exp[3yn \ln(y)]. \quad (15)$$

Further, by Stirling's formula the total number of sets  $\mathcal{Y} \subset V$  of size  $yn$  is

$$\binom{n}{yn} \leq \exp[yn(1 - \ln y)] \leq \exp(-2yn \ln y). \quad (16)$$

Combining (15) and (16) with the union bound, we obtain

$$P[\exists \mathcal{Y} \subset V : |\mathcal{Y}| = yn, e(\mathcal{Y}) \geq \ell |\mathcal{Y}|/2] \leq \exp[yn \ln(y)].$$

Taking the union bound over all possible sizes  $yn$  completes the proof.  $\square$

*Proof of Proportion 3.5.* By Proposition 3.4 w.h.p. all  $k$ -colorings  $\sigma$  of the random graph  $G'(n, m)$  satisfy  $|\sigma^{-1}(i)| = (1 + o_k(1))n/k$  for all  $i \in [k]$ . Let us call such a  $k$ -coloring  $\sigma$  of  $G = G'(n, m)$  *good* if it has the following two properties (and *bad* otherwise):

**G1.** Step **CR1** applied to  $G, \sigma$  yields sets  $W_1, \dots, W_k, W$  that satisfy the three properties in Lemma 5.1.

**G2.** The set  $U$  created in step **CR2** has size  $|U| > \frac{n \ln \ln k}{k \ln k}$ .

Let  $Z_{\text{bad}}$  be the number of bad  $k$ -colorings of  $G'(n, m)$ . Since  $G'(\sigma)$  is just the random graph  $G'(n, m)$  conditional on  $\sigma$  being a  $k$ -coloring, we have

$$\begin{aligned} E[Z_{\text{bad}}] &= \sum_{\sigma} P[\sigma \text{ is a } k\text{-coloring of } G(n, m)] \cdot P[\sigma \text{ is bad in } G'(n, m) | \sigma \text{ is a } k\text{-coloring}] \\ &= \sum_{\sigma} P[\sigma \text{ is a } k\text{-coloring of } G(n, m)] \cdot P[\sigma \text{ is bad in } G'(\sigma)] \\ &\leq 2 \exp(-10n/k) \cdot \sum_{\sigma} P[\sigma \text{ is a } k\text{-coloring of } G(n, m)] \quad [\text{by Lemmas 5.1 and 5.2}] \\ &\leq 2 \exp(-10n/k) \cdot \exp(cn/k + o(n)) \quad [\text{by Proposition 3.4}] \\ &\leq \exp(-6n/k + o(n)) = o(1). \quad [\text{as } c \leq 4] \end{aligned}$$

Hence, w.h.p. the random graph  $G'(n, m)$  does not have a bad  $k$ -coloring.

Now, consider a good  $k$ -coloring  $\sigma$ . By Lemma 5.3, we may assume that (13) holds. To bound the size of the set  $Y$  created by step **CR3**, observe that each vertex that is added to  $Y$  contributes  $\ell$  extra edges to the subgraph spanned by  $Y$ . Thus, assume that  $|Y| > \frac{2n \ln \ln k}{k \ln k}$  and consider the first time step **CR3** has got a set  $Y'$  of size  $\lceil \frac{2n \ln \ln k}{k \ln k} \rceil$ . Then  $Y'$  spans at least  $y\ell n/2$  edges, in contradiction to (13). Hence, w.h.p.  $G'(n, m)$  is such that

$$\text{for any good } k\text{-coloring the set } Y \text{ constructed by } \mathbf{CR3} \text{ has size at most } \frac{2n \ln \ln k}{k \ln k}. \quad (17)$$

If (17) is true and  $G'(n, m)$  does not have a bad  $k$ -coloring, then for any  $k$ -coloring  $\sigma$  the set  $W \cup Y$  constructed by **CR1–CR3** has size at most  $|W \cup Y| \leq k^{-0.7}n + \frac{2n \ln \ln k}{k \ln k} \leq nk^{-2/3}$  (the bound on  $|W|$  follows from **G1**). This shows the first property asserted in Proposition 3.5, because the construction **CR1–CR3** ensures that the cover  $\hat{\sigma}$  obtained from  $\sigma$  via the whitening process **WH1–WH2** satisfies  $\hat{\sigma}(v) = \sigma(v)$  for all  $v \in V \setminus (W \cup Y)$ . By the same token, the second assertion follows because by **G1** and (17) for every color  $i \in [k]$  we have

$$|\sigma^{-1}(i) \cap (W \cup Y)| \leq |W_i| + |Y| = n \cdot o_k(1/k).$$

Finally, the **G1** and (17) also imply that there cannot be more than  $\ln^9 k$  indices  $i \in [k]$  such that  $|\sigma^{-1}(i) - n/k| > n/(k \ln^3 k)$ , which is the third assertion.  $\square$

*Proof of Corollary 1.2.* We claim that the vertices in  $F = V \setminus (W \cup Y)$  are  $\delta$ -frozen w.h.p. for  $\delta = 1/(k \ln k)$ . Indeed, assume that  $\tau$  is another  $k$ -coloring such that the set  $\Delta = \{v \in F : \tau(v) \neq \sigma(v)\}$  has size  $0 < |\Delta| < \delta n$ . Every vertex  $v \in \Delta$  has at least  $\ell$  neighbors in  $\Delta$ . Indeed, the construction **CR1–CR3** ensures that every vertex  $v \in \Delta$  has at least  $\ell$  neighbors colored  $\tau(v) \neq \sigma(v)$  in  $\sigma$ . Because  $\tau$  is a  $k$ -coloring, all of these neighbors must belong to  $\Delta$  as well. Hence,  $\Delta$  violates (13). Thus, the assertion follows from Lemma 5.3.  $\square$

## 5.2 Proof of Lemma 5.1

We begin by estimating the number of edges between different color classes. Recall that  $V_i = \sigma^{-1}(i)$  for  $i \in [k]$ , and that we are assuming that  $|V_i| = (1 + o_k(1))n/k$ . Let  $\nu_i = |V_i|$  for  $i = 1, \dots, k$ .

**Lemma 5.4** *In  $G'(\sigma)$  we have*

$$\begin{aligned} \mathbb{P} \left[ \min_{1 \leq i < j \leq k} e(V_i, V_j) \leq \frac{0.99dn}{k^2} \right] &\leq \exp(-11n/k) \quad \text{and} \\ \mathbb{P} \left[ \max_{1 \leq i < j \leq k} e(V_i, V_j) \geq \frac{1.01dn}{k^2} \right] &\leq \exp(-11n/k). \end{aligned}$$

*Proof.* Because the edges  $e_1, \dots, e_m$  are chosen independently, for any pair  $1 \leq i < j \leq k$  the random variable  $e(V_i, V_j)$  has a binomial distribution  $\text{Bin}(m, q_{ij})$ , where

$$q_{ij} = \frac{2\nu_i\nu_j}{n^2 - \sum_{l=1}^k \nu_l^2} \geq \frac{2\nu_i\nu_j}{n^2}.$$

Since we are assuming that  $\nu_i, \nu_j = (1 + o_k(1))\frac{n}{k}$ , we have  $q_{ij} \geq (2 + o_k(1))/k^2$ . Thus,  $\mathbb{E}[e(V_i, V_j)] = mq_{ij} \geq (1 + o_k(1))dn/k^2$ . Hence, the Chernoff bound yields

$$\mathbb{P} \left[ e(V_i, V_j) \leq \frac{0.99dn}{2k^2} \right] \leq \exp \left[ -\frac{dn}{8 \cdot 10^4 k^2} \right] \leq \exp(-12n/k).$$

Finally, the first assertion follows by taking a union bound over  $i, j$ . The second assertion follows analogously.  $\square$

*Proof of Lemma 5.1.* By Lemma 5.4 we may disregard the case that  $\min_{1 \leq i < j \leq k} e(V_i, V_j) \leq \frac{0.99dn}{k^2}$ . Thus, fix integers  $(m_{ij})_{1 \leq i < j \leq k}$  such that

$$m_{ij} \geq \frac{0.99dn}{k^2} \quad \text{and} \quad \sum_{1 \leq i < j \leq k} m_{ij} = m. \quad (18)$$

Let  $\mathcal{M}$  be the event that  $e(V_i, V_j) = m_{ij}$  for all  $1 \leq i < j \leq k$ .

We need to get a handle on the random variables  $(e(v, V_j))_{v \in V_i}$  (i.e., the number of neighbors of  $v$  in  $V_j$ ) in the random graph  $G'(\sigma)$ . Given that  $\mathcal{M}$  occurs we know that  $\sum_{v \in V_i} e(v, V_j) = e(V_i, V_j) = m_{ij}$ . Furthermore, because  $G'(\sigma)$  consists of  $m$  independent random edges  $e_1, \dots, e_m$ , given the event  $\mathcal{M}$  the  $m_{ij}$  edges between  $V_i$  and  $V_j$  are chosen uniformly and independently. Therefore, we can think of the vertices in  $V_i$  as “bins” and of the  $m_{ij}$  edges as randomly tossed “balls”. In particular, the average number of balls that each bin  $v \in V_i$  receives is  $m_{ij}/\nu_i$ . Crucially, these balls-and-bins experiments are independent for all  $i, j$ .

To analyze them, we are going to use Corollary 2.4. Thus, consider a family  $(b_{vj})_{v \in V_i, j \in [k] \setminus \{\sigma(v)\}}$  of mutually independent Poisson variables with means  $\mathbb{E}[b_{vj}] = m_{\sigma(v)j}/\nu_{\sigma(v)}$ . Then for any family  $(t_{vj})_{v \in V_i, j \in [k] \setminus \{\sigma(v)\}}$  of integers we have

$$\mathbb{P}[\forall v, j : e(v, V_j) = t_{vj} | \mathcal{M}] \leq \exp(o(n)) \cdot \mathbb{P}[\forall v, j : b_{vj} = t_{vj}]. \quad (19)$$

In words, the joint probability that the random variables  $(e(v, V_j))_{v \in V_i, j \in [k] \setminus \{\sigma(v)\}}$  take certain values given that  $\mathcal{M}$  occurs is dominated by the corresponding event for the random variables  $(b_{vj})$ .

If  $|W_i| > \frac{n \ln \ln k}{k \ln k}$ , then there are at least  $N = \frac{n \ln \ln k}{k \ln k}$  vertices  $v \in V_i$  such that  $\min_{j \in [k] \setminus \{i\}} e(v, V_j) < 3\ell$ . Thus, let  $\mathcal{W}_i$  be the number of vertices  $v \in V_i$  such that  $\min_{j \in [k] \setminus \{i\}} b_{vj} < 3\ell$ . Then (19) yields

$$\mathbb{P}[|W_i| \geq N | \mathcal{M}] \leq \exp(o(n)) \cdot \mathbb{P}[\mathcal{W}_i \geq N]. \quad (20)$$

Furthermore, because the random variables  $(b_{vj})_{v \in V_i, j \in [k] \setminus \{i\}}$  are mutually independent,  $\mathcal{W}_i$  is a binomial random variable with mean  $\mathbb{E}[\mathcal{W}_i] \leq \nu_i q_i$ , where  $q_i = \sum_{j \in [k] \setminus \{i\}} \mathbb{P}[\text{Po}(m_{ij}/\nu_i) \leq 3\ell]$ . Since  $\nu_i = (1 + o_k(1))n/k$  and  $m_{ij} \geq 0.99dn/k^2$ , we have  $m_{ij}/\nu_i \geq 0.98d/k \geq 1.95 \ln k$ . Recalling that  $\ell = \exp(-7) \ln k$ , we find  $\mathbb{P}[\text{Po}(m_{ij}/\nu_i) \leq 3\ell] \leq k^{-1.9}$  and thus  $q_i \leq (k-1)k^{-1.9}$ . Hence,

$$\mathbb{E}[\mathcal{W}_i] \leq (1 + o_k(1))k^{-1.9}n \leq k^{-1.8}n. \quad (21)$$

Therefore, the Chernoff bound gives

$$\mathbb{P}[\mathcal{W}_i \geq N] \leq \exp \left[ -N \ln \left( \frac{k^{1.8}N}{en} \right) \right] \leq \exp(-20n/k). \quad (22)$$

Combining (20) and (22), we obtain

$$\mathbb{P}[|W_i| \geq N | \mathcal{M}] \leq \exp(o(n) - 20n/k) \leq \exp(-19n/k). \quad (23)$$

Now, consider the event that there are at least  $\kappa = \lceil \ln^4 k \rceil$  classes  $i_1, \dots, i_\kappa$  such that  $|W_{i_\ell}| \geq N' = \frac{n}{k \ln^4 k}$ . We have

$$\mathbb{P}[\mathcal{W}_{i_j} \geq N'] \leq \exp \left[ -N' \ln \left( \frac{k^{1.8}N'}{en} \right) \right] \leq \exp \left[ -\frac{1}{2}N' \ln k \right], \quad (24)$$

Furthermore, because the random variables  $\mathcal{W}_{i_1}, \dots, \mathcal{W}_{i_\kappa}$  are independent, we obtain from (19) and (24)

$$\mathbb{P}[|\{i \in [k] : |W_i| \geq N'\}| \geq \kappa | \mathcal{M}] \leq \binom{k}{\kappa} \exp \left[ -\frac{\kappa}{2}N' \ln k \right] \leq \exp(-20n/k). \quad (25)$$

With respect to the event  $|W| \geq nk^{-0.7}$ , observe that by (21) the sum  $\mathcal{W} = \sum_{i=1}^k \mathcal{W}_i$  is stochastically dominated by a binomial random variable with mean  $nk^{-0.8}$ . Therefore, by (19) and the Chernoff bound

$$\mathbb{P}[|W| \geq nk^{-0.7} | \mathcal{M}] \leq \mathbb{P}[\mathcal{W} \geq nk^{-0.7}] \leq \exp(-nk^{-0.7}) \leq \exp(-20n/k). \quad (26)$$

Finally, since the estimates (23), (25), (26) hold for all  $\mathcal{M}$ , the assertion follows from Bayes' formula.  $\square$

### 5.3 Proof of Lemma 5.2

We begin by estimating the number of edges between the sets  $W_i$  and the color class  $V_j$ . As before, we let that  $V_i = \sigma^{-1}(i)$  for  $i \in [k]$  and  $\nu_i = |V_i| = (1 + o_k(1))n/k$  for  $i = 1, \dots, k$ .

**Lemma 5.5** *In  $G'(\sigma)$  we have*

$$\mathbb{P} \left[ \max_{1 \leq i < j \leq k} e(W_i, V_j) \geq \frac{2n \ln \ln k}{k} \right] \leq \exp(-11n/k).$$

*Proof.* Fix  $1 \leq i < j \leq k$ . We begin by proving the following statement.

$$\text{For any set } S \subset V_i \text{ of size } |S| \leq \frac{n \ln \ln k}{k \ln k} \text{ we have } \mathbb{P}[e(S, V_j) > 2n \ln \ln k / k] \leq \exp(-13n/k). \quad (27)$$

Indeed, for any set  $S$  as above the number  $e(S, V_j)$  of edges  $e_i$  that join  $S$  to  $V_j$  has a binomial distribution  $\text{Bin}(m, q_{j,S})$ , where

$$q_{j,S} = \frac{2\nu_j |S|}{n^2 - \sum_{l=1}^k \nu_l^2} \leq \frac{3 \ln \ln k}{k^2 \ln k};$$

the last inequality follows from our assumption that  $\nu_l = (1 + o_k(1))n/k$  for all  $l \in [k]$ . Hence,

$$\mathbb{E}[e(S, V_j)] = mq_{j,S} \leq \frac{3d \ln \ln k}{2k^2 \ln k} \cdot n \leq \frac{3 \ln \ln k}{2k} \cdot n$$

Thus, (27) follows from the Chernoff bound. Taking the union bound over all possible sets  $S$  of size  $|S| \leq \frac{n \ln \ln k}{k \ln k}$ , we obtain from (27)

$$\mathbb{P} \left[ \exists S \subset V_i, |S| \leq \frac{n \ln \ln k}{k \ln k} : e(S, V_j) > \frac{2n \ln \ln k}{k} \right] \leq 2^{\nu_i} \exp(-13n/k) \leq \exp(-12n/k). \quad (28)$$

As  $\mathbb{P}[|W_i| > \frac{n \ln \ln k}{k \ln k}] \leq \exp(-16n/k)$  by Lemma 5.1, the assertion follows from (28).  $\square$

**Lemma 5.6** *Let  $T_i$  be the number of vertices  $v \in V_i$  such that  $\max_{j \neq i} e(v, V_j) > 100 \ln k$  and let  $T = \sum_{i \in [k]} T_i$ . Then in  $G'(\sigma)$  we have*

$$\mathbb{P} \left[ T > \frac{n}{4k \ln k} \right] \leq \exp(-10n/k).$$

*Proof.* For an integer vector  $\mathbf{m} = (m_{ij})_{1 \leq i < j \leq k}$  let  $\mathcal{E}_{\mathbf{m}}$  be the event that  $e(V_i, V_j) = m_{ij}$  for all  $1 \leq i < j \leq k$ . Set  $m_{ji} = m_{ij}$  for  $1 \leq i < j \leq k$ . By Lemma 5.4 we may confine ourselves to the case that  $e(V_i, V_j) \leq \frac{2dn}{k^2}$  for all  $i \neq j$ . Thus, fix any  $\mathbf{m}$  such that  $m_{ij} \leq \frac{2dn}{k^2}$  for all  $i < j$ . Given  $\mathcal{E}_{\mathbf{m}}$ , for each of the  $m_{ij}$  edges between color classes  $V_i, V_j$  the actual vertex in  $V_i$  that the edge is incident with is uniformly distributed. Thus, we can think of the vertices  $v \in V_i$  as bins and of edge  $m_{ij}$  edges as balls of color  $j$ , and our goal is to figure out the probability that bin  $v$  contains more than  $100 \ln k$  balls colored  $j$  for some  $j \neq i$ . Because we are conditioning on  $\mathcal{E}_{\mathbf{m}}$ , these balls-and-bins experiments are independent for all color pairs  $i \neq j$ .

To study these balls-and-bins experiments we use Corollary 2.4. Thus, let  $(b_{vj})_{v \in V, j \in [k] \setminus \{\sigma(v)\}}$  be a family of mutually independent Poisson variables such that  $\mathbb{E}[b_{vj}] = m_{ij}/\nu_i$  for all  $v \in V_i, j \in [k] \setminus \{i\}$ . In addition, let  $\mathcal{T}_i$  be the number of vertices  $v \in V_i$  such that  $\max_{j \neq i} b_{vj} > 100 \ln k$  and let  $\mathcal{T} = \sum_{i=1}^k \mathcal{T}_i$ . Then Corollary 2.4 gives

$$\mathbb{P} \left[ T > \frac{n}{4k \ln k} \mid \mathcal{E}_{\mathbf{m}} \right] \leq \exp(o(n)) \cdot \mathbb{P} \left[ \mathcal{T} > \frac{n}{4k \ln k} \right] \quad (29)$$

To complete the proof we need to bound  $\mathbb{P} \left[ \mathcal{T} > \frac{n}{4k \ln k} \right]$ . For each vertex  $v \in V_i$  and each  $j \neq i$  we have  $\mathbb{E}[b_{vj}] = m_{ij}/\nu_i \leq \frac{2dn}{k^2 \nu_i} \leq 3 \ln k$ . Hence, by Stirling's formula

$$\mathbb{P}[b_{vj} > 100 \ln k] \leq \sum_{s > 100 \ln k} \mathbb{E}[b_{vj}]^s / s! \leq k^{-90}.$$

Because the random variables  $b_{vj}$  are mutually independent,  $\mathcal{T}$  is a sum of independent Bernoulli random variables. Applying the union bound, we thus have

$$\mathbb{P} \left[ \max_{j \neq \sigma(v)} b_{vj} > 100 \ln k \right] \leq k^{-89} \quad \text{for any } v \in V. \quad (30)$$

Therefore, (30) shows that  $\mathcal{T}$  is stochastically dominated by a binomial random variable  $\text{Bin}(n, k^{-89})$ . Consequently, the Chernoff bound yields

$$\mathbb{P} \left[ \mathcal{T} > \frac{n}{4k \ln k} \right] \leq \mathbb{P} \left[ \text{Bin}(n, k^{-89}) > \frac{n}{4k \ln k} \right] \leq \exp(-20n/k). \quad (31)$$

Finally, combining (29) and (31) yields the assertion.  $\square$

*Proof of Lemma 5.2.* Let  $\mathbf{d} = (d_{vj})_{v \in V, j \in [k] \setminus \{\sigma(v)\}}$  be an integer vector. Moreover, let  $\mathcal{E}_{\mathbf{d}}$  be the event that  $e(v, V_j) = d_{vj}$  for all  $v \in V, j \neq \sigma(v)$ . We are going to estimate the size of  $U$  given that  $\mathcal{E}_{\mathbf{d}}$  occurs for a vector  $\mathbf{d}$  that is “compatible” with the properties established in Lemmas 5.4–5.6. More precisely, we call  $\mathbf{d}$  *feasible* if the following conditions are satisfied.

- i. For all  $i \neq j$  we have  $m_{ij} = \sum_{v \in V_i} d_{vj} \geq \frac{dn}{2k^2}$ . Moreover,  $m_{ij} = m_{ji}$ .
- ii. For all  $i \neq j$  we have  $w_{ij} = \sum_{v \in V_i: d_{vj} \leq 3\ell} d_{vj} \leq \frac{2n \ln \ln k}{k}$ .
- iii. Let  $\mathcal{T}$  be the set of all vertices  $v$  such that  $\max_{j \neq \sigma(v)} d_{vj} > 100 \ln k$ . Then  $|\mathcal{T}| \leq \frac{n}{4k \ln k}$ .

By Lemmas 5.4–5.6, we just need to show that for any feasible  $\mathbf{d}$  we have

$$\mathbb{P} \left[ |U| > \frac{n \ln \ln k}{k \ln k} \mid \mathcal{E}_{\mathbf{d}} \right] \leq \exp(-10n/k). \quad (32)$$

Given the event  $\mathcal{E}_{\mathbf{d}}$ , the total number  $m_{ij}$  of  $V_i$ - $V_j$ -edges is fixed. So is the number  $w_{ji}$  of  $W_j$ - $V_i$  edges. What remains random is how these edges are matched to the vertices in  $V_i$ . More specifically, think of the  $W_j$ - $V_i$ -edges as black balls, of the  $V_j \setminus W_j$ - $V_i$ -edges as white balls, and of the vertices  $v \in V_i$  as bins. Each bin  $v$  has a capacity  $d_{vj}$ . Now, the balls are tossed randomly into the bins, and our objective is to figure out the number of bins that receive more than  $\ell$  black balls. Observe that these numbers are independent for all pairs  $i \neq j$  of colors.

To estimate this probability, consider a family  $(b_{vj})_{v \in V, j \in [k] \setminus \{\sigma(v)\}}$  of independent binomial random variables such that  $b_{vj}$  has distribution  $\text{Bin}(d_{vj}, w_{ji}/m_{ji})$ . Let  $\mathcal{B}$  be the event that  $\sum_{v \in V_i} b_{vj} = w_{ji}$  for all  $i \neq j$ . Furthermore, let  $\mathcal{U}$  be the number of vertices  $v$  such that  $\max_{j \neq \sigma(v)} b_{vj} > \ell$ . Then

$$\mathbb{P} \left[ |U| > \frac{n \ln \ln k}{k \ln k} \mid \mathcal{E}_{\mathbf{d}} \right] = \mathbb{P} \left[ \mathcal{U} > \frac{n \ln \ln k}{k \ln k} \mid \mathcal{B} \right] \leq \frac{\mathbb{P} [\mathcal{U} > \frac{n \ln \ln k}{k \ln k}]}{\mathbb{P} [\mathcal{B}]}. \quad (33)$$

The sums  $\sum_{v \in V_i} b_{vj}$  are binomial random variables  $\text{Bin}(m_{ij}, w_{ji}/m_{ji})$ . Moreover, they are independent for all  $i \neq j$ . Therefore, Stirling’s formula yields

$$\mathbb{P} [\mathcal{B}] = \prod_{i \neq j} \mathbb{P} [\text{Bin}(m_{ij}, w_{ji}/m_{ji}) = w_{ji}] = \Theta(n^{-k(k-1)/2}) = \exp(o(n)). \quad (34)$$

Let  $v \in V_i$  be a vertex such that for color  $j \neq i$  we have  $d_{vj} \leq 100 \ln k$ . Then our assumptions i. and ii. on  $\mathbf{d}$  ensure that  $\mathbb{E} [b_{vj}] = \frac{w_{ji} d_{vj}}{m_{ji}} \leq 300 \ln \ln k$ . Therefore, by the Chernoff bound

$$\mathbb{P} [b_{vj} \geq \ell] \leq \exp \left[ -\ell \cdot \ln \left( \frac{\ell}{e \cdot \mathbb{E} [b_{vj}]} \right) \right] \leq k^{-100}.$$

Hence, taking the union bound, we find

$$\mathbb{P} \left[ \max_{j \neq \sigma(v)} b_{vj} \geq \ell \right] \leq k^{-99} \quad \text{if } \max_{j \neq \sigma(v)} d_{vj} \leq 100 \ln k. \quad (35)$$

Let  $\mathcal{U}'$  be the number of vertices  $v$  such that  $\max_{j \neq \sigma(v)} b_{vj} \geq \ell$  while  $\max_{j \neq \sigma(v)} d_{vj} \leq 100 \ln k$ . Because the random variables  $b_{vj}$  are independent, (35) implies that  $\mathcal{U}'$  is stochastically dominated by a binomial random variable  $\text{Bin}(n, k^{-99})$ . Therefore, the Chernoff bound gives

$$\mathbb{P} \left[ \mathcal{U}' \geq \frac{n \ln \ln k}{2k \ln k} \right] \leq \mathbb{P} \left[ \text{Bin}(n, k^{-99}) \geq \frac{n \ln \ln k}{2k \ln k} \right] \leq \exp(-11n/k). \quad (36)$$

As  $\mathcal{U} \leq \mathcal{T} + \mathcal{U}' \leq \mathcal{U}' + \frac{n}{4k \ln k}$  by our assumption iii. on  $\mathbf{d}$ , (36) implies that  $\mathbb{P} \left[ \mathcal{U} \geq \frac{n \ln \ln k}{k \ln k} \right] \leq \exp(-11n/k)$ . Thus, the assertion follows from (33) and (35).  $\square$

## 6 Proof of Proposition 3.6

Throughout this section, we let  $\zeta : V \rightarrow \{0, 1, \dots, k\}$ ,  $V_i = \zeta^{-1}(i)$  and  $\nu_i = |V_i|$  for  $i = 0, 1, \dots, k$ . In addition, we let  $\alpha_i = \nu_i/n$ . We always assume that the conditions of Proposition 3.6 hold, namely

**Z1.**  $|\zeta^{-1}(0)| \leq nk^{-2/3}$ .

**Z2.**  $|\zeta^{-1}(i)| = (1 + o_k(1))n/k$  for all  $i \in [k]$ .

**Z3.** There are no more than  $\ln^9 k$  indices  $i \in [k]$  such that  $|\zeta^{-1}(i) - n/k| > n/(k \ln^3 k)$ .

In addition, we assume that  $d = 2k \ln k - \ln k - c$  for some  $0 \leq c \leq 4$ .

### 6.1 Counting covers

To prove Proposition 3.6 we perform a first moment argument over the number of covers  $\zeta$ . Let  $\mathcal{I}_\zeta$  be the event that  $V_1, \dots, V_k$  are independent sets in  $G'(n, m)$ . Moreover, let  $\mathcal{C}_\zeta$  be the event that  $\zeta$  is a  $k$ -cover in  $G'(n, m)$ . Clearly,  $\mathcal{C}_\zeta \subset \mathcal{I}_\zeta$ , and we begin by estimating the probability the latter event. Let  $F_\zeta = \sum_{j=1}^k \alpha_j^2$ .

**Lemma 6.1** *We have  $\frac{1}{n} \ln \mathbb{P}[\mathcal{I}_\zeta] = \frac{d}{2} \ln(1 - F_\zeta)$ .*

*Proof.* For each of the edges  $e_i$  the probability of joining two vertices in  $V_j$  is  $(\nu_j/n)^2 = \alpha_j^2$ . Hence, the probability that  $e_i$  does not fall inside any of the classes  $V_1, \dots, V_k$  is equal to  $1 - F_\zeta$ . Thus, the assertion follows from the independence of  $e_1, \dots, e_m$ .  $\square$

In Section 6.2 we are going to establish the following estimate of the probability of  $\mathcal{C}_\sigma$ .

**Lemma 6.2** *We have  $\frac{1}{n} \ln \mathbb{P}[\mathcal{C}_\zeta | \mathcal{I}_\zeta] \leq \sum_{i=0}^k \alpha_i \ln p_i + o(1)$ , where*

$$\begin{aligned} p_0 &= \sum_{i,j \in [k]: i \neq j} \left( \frac{1}{2} + \frac{\alpha_j d}{1 - F_\zeta} \right) \exp \left[ -\frac{(\alpha_i + \alpha_j)d}{1 - F_\zeta} \right], \\ p_i &= \prod_{j \in [k] \setminus \{i\}} 1 - \left( 1 + \frac{\alpha_j d}{1 - F_\zeta} \right) \exp \left[ -\frac{\alpha_j d}{1 - F_\zeta} \right] \quad \text{for } i = 1, \dots, k. \end{aligned}$$

*Proof of Proposition 3.6.* Let  $A$  be the set of all vectors  $\alpha = (\alpha_0, \dots, \alpha_k) \in [0, 1]^{k+1}$  that satisfy the following three conditions (cf. **Z1–Z3**):

- i.  $\sum_{i=0}^k \alpha_i = 1$ ,
- ii. We have  $\alpha_0 \leq k^{-2/3}$  and  $\alpha_i = (1 + o_k(1))/k$  for  $i = 1, \dots, k$ . Indeed, there are no more than  $K = \ln^9 k$  indices  $i \in [k]$  such that  $|\alpha_i - 1/k| > k^{-1} \ln^{-3} k$ .
- iii.  $\alpha_i n$  is an integer for  $i = 0, 1, \dots, k$ .



For  $\alpha \in A$  let  $\mathcal{S}_\alpha$  be the set of all maps  $\zeta : V \rightarrow \{0, 1, \dots, k\}$  such that  $|\zeta^{-1}(i)| = \alpha_i n$  for all  $i$ . Then

$$\mathcal{S}_\alpha = \binom{n}{\alpha_0 n, \dots, \alpha_k n} = \binom{n}{\alpha_0 n} \cdot \binom{(1 - \alpha_0)n}{\alpha_1 n, \dots, \alpha_k n} \leq \binom{n}{\alpha_0 n} \cdot k^{(1 - \alpha_0)n}.$$

Hence, Stirling's formula yields

$$\frac{1}{n} \ln \mathcal{S}_\alpha \leq -\alpha_0 \ln \alpha_0 - (1 - \alpha_0) \ln((1 - \alpha_0)/k). \quad (37)$$

Lemmas 6.1 and 6.2 show that for any  $\zeta \in \mathcal{S}_\alpha$ ,

$$\frac{1}{n} \ln P[\mathcal{C}_\zeta] \leq o(1) + \frac{d}{2} \ln(1 - F_\zeta) + \sum_{i=0}^k \alpha_i \ln p_i.$$

Given the value of  $\alpha_0$ , the sum  $F_\zeta = \sum_{i=1}^k \alpha_i^2$  is minimized if  $\alpha_i = (1 - \alpha_0)/k$  for all  $i \in [k]$ . Thus,

$$\frac{1}{n} \ln P[\mathcal{C}_\zeta] \leq o(1) + \frac{d}{2} \ln(1 - (1 - \alpha_0)^2/k) + \sum_{i=0}^k \alpha_i \ln p_i. \quad (38)$$

Using the approximation  $\ln(1 - z) = -z - z^2/2 + O(z^3)$  and recalling that  $d = 2k \ln k - \ln k - c$ , we see that

$$\begin{aligned} \frac{d}{2} \ln(1 - (1 - \alpha_0)^2/k) &= -(1 - \alpha_0)^2 \ln k \\ &\quad + \frac{(1 - \alpha_0)^2 \ln k}{2k} + \frac{c(1 - \alpha_0)^2}{2k} - \frac{(1 - \alpha_0)^4 \ln k}{2k} + O_k(k^{-1.9}) \\ &= -(1 - 2\alpha_0) \ln k + \frac{c}{2k} + o_k(1/k) \quad [\text{as } \alpha_0 \leq k^{-2/3} \text{ by ii.}] \end{aligned} \quad (39)$$

Furthermore, because  $F_\zeta \in (0, 1)$  and as  $\ln(1 - z) \leq -z$  for all  $z \in (0, 1)$ , we get

$$\begin{aligned} \sum_{i=1}^k \alpha_i \ln p_i &\leq - \sum_{i,j \in [k]: i \neq j} \alpha_i (1 + \alpha_j d) \exp(-\alpha_j d) \\ &= - \sum_{j \in [k]} (1 - \alpha_0 - \alpha_j) (1 + \alpha_j d) \exp(-\alpha_j d). \end{aligned} \quad (40)$$

Since  $\alpha_j = (1 + o_k(1))/k$  for all  $j \in [k]$  by ii. and as  $d = 2k \ln k - O_k(\ln k)$ , (40) yields

$$\sum_{i=1}^k \alpha_i \ln p_i \leq O_k(k^{-1.9}) - (1 - \alpha_0) \sum_{j \in [k]} (1 + \alpha_j d) \exp(-2\alpha_j k \ln k). \quad (41)$$

Moreover, applying condition ii., we obtain from (41)

$$\begin{aligned} \sum_{i=1}^k \alpha_i \ln p_i &\leq O_k(k^{-1.9}) + O(K \ln k) \cdot \exp(-2(1 + o_k(1)) \ln k) \\ &\quad - (1 - \alpha_0)(k - K)(1 + 2 \ln k + O_k(1/\ln^2 k)) \exp(-2(1 + O_k(\ln^{-3} k)) \ln k) \\ &\leq o_k(1/k) - (1 - \alpha_0) \cdot \frac{1 + 2 \ln k}{k} \quad [\text{as } K \leq k^{0.01}] \\ &\leq o_k(1/k) - \frac{1 + 2 \ln k}{k} \quad [\text{as } \alpha_0 \leq k^{-2/3} \text{ by ii.}] \end{aligned} \quad (42)$$

Further, again because  $F_\zeta \in (0, 1)$  we have

$$\begin{aligned} p_0 &\leq \frac{1}{2} \sum_{i,j \in [k]: i \neq j} (1 + 2\alpha_j d) \exp[-(\alpha_i + \alpha_j)d], \\ &\leq O_k(k^{-3+o_k(1)} K) + \frac{k(k-1)}{2} [1 + 4 \ln k + O_k(\ln^{-2} k)] \exp[-4 \ln k + O(\ln^{-2} k)] \quad [\text{by condition ii.}] \\ &\leq \frac{1 + 4 \ln k + O_k(\ln^{-1} k)}{2k^2}. \end{aligned}$$

Hence,

$$\alpha_0 \ln p_0 \leq \alpha_0 \ln \left( \frac{1 + 4 \ln k}{2k^2} \right) + \alpha_0 \cdot o_k(1). \quad (43)$$

Plugging (39), (42) and (43) into (38), we obtain

$$\begin{aligned} \frac{1}{n} \ln P[\mathcal{C}_\zeta] &\leq \frac{c}{2k} - (1 - 2\alpha_0) \ln k - \frac{1 + 2 \ln k}{k} + \alpha_0 \ln \left( \frac{1 + 4 \ln k}{2k^2} \right) + \alpha_0 \cdot o_k(1) + o_k(1/k) \\ &= \frac{c - 2 - 4 \ln k}{2k} - \ln k + \alpha_0 \ln \left( \frac{1 + 4 \ln k}{2} \right) + \alpha_0 \cdot o_k(1) + o_k(1/k). \end{aligned} \quad (44)$$

Finally, combining (37) and (44), we get

$$\begin{aligned} \frac{1}{n} \ln(|\mathcal{S}_\alpha| \cdot P[\mathcal{C}_\zeta]) &\leq \frac{c - 2 - 4 \ln k}{2k} - \alpha_0 \ln \left( \frac{2k\alpha_0}{1 + 4 \ln k} \right) \\ &\quad - (1 - \alpha_0) \ln(1 - \alpha_0) + \alpha_0 \cdot o_k(1) + o_k(1/k) \\ &\leq \frac{c - 2 - 4 \ln k}{2k} + \alpha_0 \left[ 1 - \ln \left( \frac{2k\alpha_0}{1 + 4 \ln k} \right) + o_k(1) \right] + o_k(1/k). \end{aligned} \quad (45)$$

Elementary calculus shows that the function  $\alpha_0 \in (0, 1) \mapsto -\alpha_0(1 - \ln \frac{2k\alpha_0}{1 + 4 \ln k} + o_k(1))$  attains its maximum at  $\alpha_0 = (1 + o_k(1)) \frac{1 + 4 \ln k}{2k}$ . Hence, (45) yields

$$\frac{1}{n} \ln(|\mathcal{S}_\alpha| \cdot P[\mathcal{C}_\zeta]) \leq \frac{c - 1 + o_k(1)}{2k}. \quad (46)$$

To complete the proof, consider for any  $\alpha \in A$  the number  $\Sigma_\alpha$  of  $k$ -covers  $\zeta$  of  $G'(n, m)$  such that  $|\zeta^{-1}(i)| = \alpha_i$  for all  $i$ . Then (46) implies that  $\frac{1}{n} \ln E[\Sigma_\alpha] \leq \frac{c-1}{2k} - o_k(1)$  for all  $\alpha \in A$ . Hence, there is  $0 < \varepsilon_k = o_k(1)$  such that for  $c < 1 - \varepsilon_k$  we have

$$E[\Sigma_\alpha] \leq \exp \left[ \frac{c-1}{2k} - o_k(1) \right] \leq \exp(-\varepsilon_k/2) = \exp(-\Omega(n)). \quad (47)$$

Since condition iii. ensures that  $|A| \leq n^k = \exp(o(n))$ , the assertion follows from (47) by taking the union bound over all  $\alpha \in A$  and applying Lemma 2.1.  $\square$

## 6.2 Proof of Lemma 6.2

Given  $\mathcal{I}_\zeta$ , the pairs  $e_1, \dots, e_m$  that constitute the random graph  $G'(n, m)$  are simply distributed uniformly and independently over the set of all  $n^2(1 - F_\zeta)$  possible pairs that do not join two vertices in the same class  $V_i$  for  $i = 1, \dots, k$ . For each vertex  $v$  and each  $j \in \{0, 1, \dots, k\}$  let  $d_{v,j}$  be the number of pairs  $e_i$  such that  $e_i$  contains  $v$  together with a vertex from  $V_j$ . Clearly, given  $\mathcal{I}_\zeta$  we have  $d_{v,j} = 0$  for all  $v \in V_j$ ,  $j \in [k]$ .

It seems reasonable to expect that the  $d_{v,j}$  are asymptotically independent Poisson random variables. To state this precisely, consider a family  $(b_{vj})_{v \in V, j \in \{0, 1, \dots, k\}}$  of independent Poisson random variables with means

$$E[b_{vj}] = \frac{\alpha_j d}{1 - F_\zeta}.$$

Let  $\mathcal{B}_\zeta$  be the event that

- i. for any  $v \in V_0$  there exist  $i, j \in [k]$ ,  $i \neq j$  such that  $b_{vi} = 0$  and  $b_{vj} \leq 1$  and
- ii. for any  $1 \leq i < j \leq k$  and any  $v \in V_i$  we have  $b_{vj} > 1$ .

The key step in the proof (somewhat reminiscent of the Poisson cloning model [25]) is to establish the following.

**Lemma 6.3** We have  $P[\mathcal{C}_\zeta | \mathcal{I}_\zeta] \leq \exp(o(n)) \cdot P[\mathcal{B}_\zeta]$ .

To prove Lemma 6.3 we consider a further event. Set  $B_{ij} = \sum_{v \in V_i} b_{vj}$  for  $i, j \in \{0, 1, \dots, k\}$ ,  $(i, j) \neq (0, 0)$ . Being sums of independent Poisson variables, the random variables  $B_{ij}$  are Poisson as well, with means

$$E[B_{ij}] = E[B_{ji}] = \alpha_i \alpha_j dn / (1 - F_\zeta) \quad (0 \leq i < j \leq k).$$

In addition, let  $B_{00}$  be a random variable that is independent of all of the above such that  $\frac{1}{2}B_{00}$  has distribution  $\text{Po}(\alpha_0^2 m / (1 - F_\zeta))$ . (In particular,  $B_{00}$  takes even values only.) Now, let  $\mathcal{V}$  be the event that

- i.  $B_{ij} = B_{ji}$  for all  $i \neq j$  and
- ii.  $\frac{1}{2}B_{00} + \sum_{0 \leq i < j \leq k} B_{ij} = m$ .

**Lemma 6.4** We have  $P[\mathcal{V}] = \exp(o(n))$ .

*Proof.* Since

$$E\left[\frac{B_{00}}{2} + \sum_{1 \leq i < j \leq k} B_{ij}\right] = \frac{dn}{2(1 - F(\zeta))} \left[\alpha_0^2 + \sum_{i \neq j} \alpha_i \alpha_j\right] = m,$$

there exist integers  $\beta_{ij} = E[B_{ij}] + O(1)$  such that  $\beta_{ij} = \beta_{ji}$  and  $\frac{1}{2}\beta_{00} + \sum_{0 \leq i < j \leq k} \beta_{ij} = m$ . Clearly,

$$P[\mathcal{V}] \geq P[B_{ij} = \beta_{ij} \text{ for all } i, j] = \prod_{i, j} P[B_{ij} = \beta_{ij}]. \quad (48)$$

Since  $\beta_{ij} = E[B_{ij}] + O(1)$  and  $B_{ij}$  is a Poisson variable, Stirling's formula yields  $P[B_{ij} = \beta_{ij}] = \Omega(n^{-1/2})$ . Therefore, (48) implies  $P[\mathcal{V}] \geq \Omega(n^{-(k+1)^2/2}) = \exp(o(n))$ , as claimed.  $\square$

*Proof of Lemma 6.3.* Let  $\mathbf{m} = (m_{ij})_{i, j \in \{0, 1, \dots, k\}}$  be a family of non-negative integers such that

- a.  $m_{ij} = m_{ji}$  for all  $i, j$ ,
- b.  $m_{ii} = 0$  for  $i \in [k]$  and
- c.  $m_{00} + \sum_{0 \leq i < j \leq k} m_{ij} = m$ .

Let  $\mathcal{M}_\mathbf{m}$  be the event that

$$\sum_{v \in V_0} d_{v0} = 2m_{00} \quad \text{and} \quad \sum_{v \in V_i} d_{vj} = m_{ij} \text{ for any } 0 \leq i < j \leq k.$$

Analogously, let  $\mathcal{M}'_\mathbf{m}$  be the event that

$$B_{00} = 2m_{00} \quad \text{and} \quad B_{ij} = m_{ij} \text{ for any } 0 \leq i < j \leq k.$$

We claim that for any  $\mathbf{m}$  that satisfies a.–c. above we have

$$P[\mathcal{C}_\zeta | \mathcal{M}_\mathbf{m}] = P[\mathcal{B}_\zeta | \mathcal{M}'_\mathbf{m}]. \quad (49)$$

Indeed, let either  $i = j = 0$  or  $0 \leq i < j \leq k$ . Given that  $\mathcal{M}_\mathbf{m}$  occurs, we can think of the  $m_{ij}$  edges that join  $V_i$  and  $V_j$  as balls and of the vertices  $v \in V_i$  as bins. Each ball is tossed into one of the bins randomly and independently, and these events are independent for all  $i, j$ . Thus, (49) simply follows from the Poissonization of the balls and bins experiment (Lemma 2.3).

To complete the proof, we need to compare  $P[\mathcal{M}_\mathbf{m} | \mathcal{I}_\zeta]$  and  $P[\mathcal{M}'_\mathbf{m} | \mathcal{V}]$ . Because under the distribution  $P[\cdot | \mathcal{I}_\zeta]$  the pairs  $e_1, \dots, e_m$  are simply chosen randomly subject to the constraint that none of them joins two vertices in the same class  $V_i$ ,  $i \in [k]$ , we see that

$$P[\mathcal{M}_\mathbf{m} | \mathcal{I}_\zeta] = \frac{m!}{m_{00}! \prod_{0 \leq i < j \leq k} m_{ij}!} \cdot \left(\frac{\alpha_0^2}{1 - F_\zeta}\right)^{m_{00}} \prod_{0 \leq i < j \leq k} \left(\frac{2\alpha_i \alpha_j}{1 - F_\zeta}\right)^{m_{ij}}. \quad (50)$$

(The factor of 2 arises because  $e_1, \dots, e_m$  are *ordered* pairs.) Furthermore, because  $\mathcal{V}$  provides that  $B_{ij} = B_{ji}$  for all  $i, j$ , we have

$$\mathbb{P}[\mathcal{M}'_{\mathbf{m}}|\mathcal{V}] = \frac{\mathbb{P}[B_{00} = 2m_{00}] \cdot \prod_{0 \leq i < j \leq k} \mathbb{P}[B_{ij} = m_{ij}]}{\mathbb{P}[\mathcal{V}]}.$$

Thus, by Lemma 6.4

$$\mathbb{P}[\mathcal{M}'_{\mathbf{m}}|\mathcal{V}] = \exp(o(n)) \cdot \mathbb{P}[B_{00} = 2m_{00}] \cdot \prod_{0 \leq i < j \leq k} \mathbb{P}[B_{ij} = m_{ij}]. \quad (51)$$

Since for  $0 \leq i < j \leq k$  the random variables  $B_{ij}$  are Poisson with mean  $\alpha_i \alpha_j dn / (1 - F_\zeta)$ , we have

$$\mathbb{P}[B_{ij} = m_{ij}] = \frac{(\alpha_i \alpha_j dn / (1 - F_\zeta))^{m_{ij}}}{m_{ij}! \exp(\alpha_i \alpha_j dn / (1 - F_\zeta))} = \left( \frac{2\alpha_i \alpha_j}{1 - F_\zeta} \right)^{m_{ij}} \frac{m_{ij}^{m_{ij}}}{m_{ij}! \exp(2\alpha_i \alpha_j m / (1 - F_\zeta))}. \quad (52)$$

Similarly,

$$\mathbb{P}[B_{00} = 2m_{00}] = \frac{(\alpha_0^2 m / (1 - F_\zeta))^{m_{00}}}{m_{00}! \exp(\alpha_0^2 m / (1 - F_\zeta))} = \left( \frac{\alpha_0^2}{1 - F_\zeta} \right)^{m_{00}} \frac{m^{m_{00}}}{m_{00}! \exp(\alpha_0^2 m / (1 - F_\zeta))}. \quad (53)$$

Combining (50)–(53), we obtain from Stirling's formula

$$\begin{aligned} \frac{\mathbb{P}[\mathcal{M}_{\mathbf{m}}|\mathcal{I}_\zeta]}{\mathbb{P}[\mathcal{M}'_{\mathbf{m}}|\mathcal{V}]} &= \frac{m! \exp(m(1 - F_\zeta)^{-1}(\alpha_0^2 + 2 \sum_{0 \leq i < j \leq k} \alpha_i \alpha_j))}{\exp(o(n)) m^m} \\ &= \frac{m! \exp(m + o(n))}{m^m} = \exp(o(n)). \end{aligned} \quad (54)$$

Finally, combining (49) and (54) we conclude that for any  $\mathbf{m}$  that satisfies a.–c. we have

$$\begin{aligned} \mathbb{P}[\mathcal{C}_\zeta \cap \mathcal{M}_{\mathbf{m}}|\mathcal{I}_\zeta] &= \mathbb{P}[\mathcal{C}_\zeta|\mathcal{M}_{\mathbf{m}}] \cdot \mathbb{P}[\mathcal{M}_{\mathbf{m}}|\mathcal{I}_\zeta] \quad [\text{as } \mathcal{M}_{\mathbf{m}} \subset \mathcal{I}_\zeta] \\ &= \exp(o(n)) \mathbb{P}[\mathcal{B}_\zeta|\mathcal{M}'_{\mathbf{m}}] \cdot \mathbb{P}[\mathcal{M}'_{\mathbf{m}}|\mathcal{V}] \\ &= \exp(o(n)) \mathbb{P}[\mathcal{B}_\zeta \cap \mathcal{M}'_{\mathbf{m}}] / \mathbb{P}[\mathcal{V}] \\ &= \exp(o(n)) \mathbb{P}[\mathcal{B}_\zeta \cap \mathcal{M}'_{\mathbf{m}}] \quad [\text{due to Lemma 6.4}]. \end{aligned}$$

Summing over all possible  $\mathbf{m}$  completes the proof.  $\square$

*Proof of Lemma 6.2.* We are going to bound the probability of the event  $\mathcal{B}_\zeta$ . For  $v \in V_0$  we have

$$\mathbb{P}[\exists 1 \leq i < j \leq k : b_{vi} = b_{vj} = 0] \leq \sum_{1 \leq i < j \leq k} \mathbb{P}[b_{vi} = b_{vj} = 0] + \sum_{i, j \in [k] : i \neq j} \mathbb{P}[b_{vi} = 0, b_{vj} = 1] = p_0,$$

because the  $b_{vi}, b_{vj}$  are independent Poisson variables. Similarly, if  $v \in V_i$  for some  $i \in [k]$ , then

$$\mathbb{P}[\forall j \in [k] \setminus \{i\} : b_{vj} > 1] = \prod_{j \in [k] \setminus \{i\}} 1 - \mathbb{P}[b_{vj} \leq 1] = p_i.$$

Due to the mutual independence of the  $b_{vj}$ , we thus obtain  $\mathbb{P}[\mathcal{B}_\zeta] = p_0^{\alpha_0 n} \prod_{i=1}^k p_i^{\alpha_i n}$ . Finally, the assertion follows from Lemma 6.3.  $\square$

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